1. Project Summary: Enhancing and Assessing Spatial Cognition through Computational Craftwork

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When professional mathematicians and scientists discuss the nature and origins of their own creativity, they often mention the central role played by visual and spatial thinking. Many of these professionals allude to the importance of visual cognition in the practice of mathematics and science; or they recall early formative experiences with spatially-rich activities such as geometric construction kits. But despite the apparent importance of visual/spatial cognition in mathematics and science education, there is still relatively little research aimed at understanding, enhancing, and assessing spatial reasoning in math and science students; and there are few efforts that seek to blend novel computational media with the sorts of spatially and mathematically rich hands-on crafting activities that influenced earlier generations of scientists. We believe that progress can be made in both these areas—i.e., basic cognitive research and pedagogical development—by an effort at integration. That is, by designing playful, creative and technologically sophisticated educational materials for exercising and enhancing spatial cognition, we can use those materials as the means to explore and understand fundamental issues in spatial cognition as well.

In the course of this project, then, we plan to:

- Characterize the nature of *spatial expertise* in the understanding of three-dimensional forms, while creating flexible and non-software-specific *assessment techniques* to measure development of that expertise;
- Devise a *practical spatial curriculum* (i.e., materials designed to enhance and exercise spatial cognition) based on a combination of hands-on work and creative computational papercrafting activities that can be adapted to a wide range of software environments;
- Extend our current software research environments to incorporate online *spatial advisors* that assist students in reflecting upon, understanding, playing with, and interpreting three-dimensional polyhedral forms; and
- Create new software to explore both traditional and novel realms of mathematical papercrafting, such as pop-ups, flexagons, anamorphic art, and surface models. Importantly, these software tools are conceived as *design environments* for students, through which they can create and print out new, beautiful, expressive, and personalized mathematical structures. Thus, rather than simply provide students with pre-existing kits or exercises, we seek to focus on developing activities that are intellectually rich and affectively compelling. In effect, then, we seek to expand the landscape of traditional mathematical papercrafts by exploiting the creative potential of computational media.

Our proposal is most directly relevant to Quadrant II of the ROLE program (Fundamental Research on Behavioral, Cognitive, Affective, and Social Aspects of Learning), in that it focuses on understanding and assessing the development of spatial cognition; our particular emphasis is on understanding the relationship between spatial cognition and hands-on craft activities in mathematical and scientific education. We expect that the project will also contribute to research in Quadrant III (Research on SMET Learning in Educational Settings), in that a major portion of our research will be directed toward the creation of flexible and practical activity-based materials for the enhancement of spatial cognition in middle and high school students.

2. Project Description

There is a tradition of folklore among mathematicians and scientists that the ability to reason "visually" is an important element—perhaps *the* most important element—of their intellectual personality. Hadamard [1949], in a classic study of mathematical thinking, noted of himself that "I insist that words are totally absent from my mind when I really think". [p. 75] He illustrated the point with an account of his understanding of a proof in number theory, accompanied by "strange and cloudy imagery". [p. 77] In a famous letter to Hadamard, Albert Einstein echoed these sentiments with his own introspection:

> The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The psychical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be "voluntarily" reproduced and combined....The above-mentioned elements are, in my case, of visual and some of muscular type. [Einstein 1954 pp. 25-26]

Likewise, the renowned mathematical physicist Stanislaw Ulam observed that

> "It is one thing to know about physics abstractly, and quite another to have a practical encounter with problems directly connected with experimental data... I found out that the main ability to have was a visual, and also an almost tactile, way to imagine the physical situations, rather than a merely logical picture of the problems... Very few mathematicians seem to possess [such an imagination] to any great degree" (quoted in [Cooper 1989], p. 15).

In an especially intriguing interview, the astronomer Margaret Geller recalled of her own childhood:

> "My father is a crystallographer.... He had an attraction for any kind of toy that had anything to do with geometry.... [T]here were toys where you could connect flat shapes up with rubber bands to make solid figures. He bought me that, and he'd explain to me the relationship between things that I built and things in the world. For example, I'd make a cube, and he'd explain to me the relationship between that and the structure of table salt. And I'd make an icosahedron, and he'd explain how you see that in the real world.... I would be able to visualize in 3-D. And I realize now--I've talked to lots of people in science--that very few people have that ability." [Lightman and Brawer 1990, p. 361]

Quotes such as these are not hard to find; collectively they suggest that spatial and visual activities, particularly when accompanied by a motor or tactile component, are especially powerful (and perhaps crucial) elements of a truly thorough mathematical and scientific education. This notion is corroborated by a half century of research in mathematical cognition and education. As early as 1948, Piaget and Inhelder reported upon experiments in which young children were asked to draw "flattened out" versions of solid shapes, and noted that "[I]magining the rotation and development of surfaces depends largely on the actual process of unfolding solids, and the motor skills involved in such actions. In particular, *the child who is familiar with folding and unfolding paper shapes through his work at school is two or three years in advance of children who lack this experience*."[p. 276; emphasis added] A similar connection was made twenty-five years later by Bishop [1973], who observed that students enrolled at schools where teaching was "based very firmly on the use of structural apparatus" outscored their peers on standardized tests of spatial thinking. Still more recently, Marjorie Senechal, in a gorgeous essay on geometric education, writes, "Hands-on experimentation is essential. For example, when we make a cube with our own hands, we gain much more insight into its metric, combinatorial, and stability properties than if we just look at one.... Ideally, shape should be taught in a laboratory setting. At the very least, every school should have a laboratory where students can explore shape."[pp. 178-9]

Beyond these observations from the realm of mathematics education, there is additional suggestive evidence from research in spatial cognition indicating that activities to strengthen students' visual and spatial reasoning should play a central role in mathematics and science education. This argument is supported by a body of cognitive science research arguing that spatial thinking is a distinct component of human intelligence [Gardner 1983]; that it is an important predictor of success in college physics courses [Siemankowski and McKnight 1971]; that visualization plays a strategic role not only in geometric but also in certain types of algebraic reasoning [Hershkowitz, Arcavi, and Bruckheimer 2001]; and, importantly, that certain types of abilities linked with spatial thinking may be taught (see, for instance, Brinkmann's [1966] report of a curriculum for teaching visual thinking, or Olson's work in developing an educational toy to help teach very young children the often-problematic concept of diagonality [Olson 1970]). In a general study of the development of spatial cognition, Newcombe and Huttenlocher [2000] conclude:

> [S]ome of the developments we have charted can be understood as the interaction of biological starting points with universally available aspects of the environment....[O]ther developments, notably in areas of symbolic spatial competence and spatial reasoning, involve aspects of the environment that show much more variability across individuals, due to the relevance of cultural transmission and specialized experiential opportunities. Developments in these areas, accordingly, are much more variable in pace and even in nature across children and cultures. [p. 208]

In our view, a coherent argument for spatial and visual education, mediated by mathematically rich craft activities, is beginning to emerge from this literature. But part of the difficulty for math and science educators is that they must proceed on the basis of a still very tentative foundation of basic research. That is, there are still many, many unresolved and fundamental questions about the nature of visual (or visuospatial) cognition—questions involving topics such as mental imagery, visual memory, the relationship between the senses of vision and touch, and the relationship between visual and linguistic cognition, to name a few. (See, for instance, [Tye 1991] and [de Vega 1996] to get a flavor of the current debates.) Still another obstacle to educational research can be found in a traditional culture of professional mathematics (and, to a lesser degree, science) that emphasizes symbolic formalisms at the expense of visual representations. As English[p.10] writes, "Despite the importance of image-based reasoning in mathematics, it has not received the attention it deserves, largely because mathematics has traditionally been viewed as a purely 'abstract' ('mind-free') discipline." In the same vein, Lakoff and Nuñez [1997] note that "Mathematics, one of the most imaginative activities in the history of humankind, has been supplied with foundations that seek not to explicate mathematical imagination precisely, but rather to ban from the characterization of mathematics all forms of imagination—images, metaphorical thought, signs, pictures, narrative forms" [p. 29; see also the fascinating book-length discussion of similar themes in Lakoff and Nuñez 2000]. Bryant and Squire [2001] describe the connection between children's spatial and mathematical understanding as "strangely neglected"; while Clements and Battista [1992], after summarizing the state of research in spatial reasoning and geometry, conclude by putting the matter succinctly: "There has been too little instructional attention given in the United States to spatial reasoning." (p. 457)

In short, then, there is an acute need for additional research in the nature of spatial cognition, and its role in mathematical understanding. But even if we accept a working definition of spatial cognition—and even if the traditional mathematical community were more receptive toward the subject—there would nonetheless remain delicate *pedagogical* questions about how to help students develop such reasoning abilities. It would certainly be possible to create "visual workbooks", or drill-and-practice software systems, focusing perhaps on the types of problems (e.g., mental-rotation tasks) typical of standardized psychological tests of visual thinking. While such efforts may prove effective in raising students' test scores, it is our belief that they would prove equally effective in destroying any sense of enjoyment, creativity, or personal expression that students might feel in their mathematical work. Indeed, as with so many other efforts in skill-training, visual drill-and-practice runs the risk of placing the cart of skill acquisition before the horse of motivation: students might well learn to perform visual tasks while becoming ever more disaffected from the context in which the learning is taking place. (Cf. much of the work of M. Csikszentmihalyi on motivation—e.g., [Csikszentmihalyi 1996, pp. 108-110].)

Our conclusion, then, is this: we believe that it is essential for cognitive research and pedagogical development to take place in the context of developing expressive, creative, dignified, and mathematically rich activities for students—activities that will support and exercise spatial skills, while serving as the means by which cognitive questions can be investigated. We propose to develop:

- Software systems for computational papercrafting, focusing on advanced mathematical, scientific, and artistic polyhedral modelling, and including activities that integrate hands-on physical materials with computer-based work;
- Experiments and research tools to investigate a variety of central questions in the development of spatial thinking in mathematics;
- Flexible curricular and assessment materials, potentially adaptable to a variety of software systems and environments (including but not limited to our own), and that teachers and researchers may use to explore students' spatial cognition.

By proceeding in this integrative fashion, we believe that cognitive research and educational development can mutually support each other: that we can explore the role of spatial cognition in mathematics even while extending an already venerable (and gorgeous!) tradition in mathematical education. [Holden, 1971; Cundy and Rollett, 1961]

Our proposed research will build upon work that we have already done in creating the *HyperGami* and *JavaGami* programs for mathematical papercrafts. Briefly, these systems permit students to create an endless variety of customized three-dimensional polyhedral forms on the computer screen; the software then "unfolds" these shapes to produce a two-dimensional folding net pattern; and finally, the net may be decorated by a variety of means, output to a color printer, and folded into a tangible model. (Much more detail may be found in [Eisenberg and Eisenberg 1999b; 1998a; Eisenberg and Nishioka 1997b].) To date, over 400 copies of HyperGami and JavaGami have been downloaded free of charge by teachers, students, and hobbyists in 38 states and 24 foreign countries; we have used the software in tutorials and classes with more than 75 elementary, middle and high school students; and the software has been used to create a variety of polyhedral sculpture kits that have been commercially available over the World Wide Web for the past year. Figure 1 shows several representative polyhedral models that have been created with the software, and many more examples—created by ourselves and our students—may be found at the HyperGami website [http://www.cs.colorado.edu/~ctg/projects/hypergami/]. Our proposed research will use our current systems as a starting point, but we will also significantly extend these systems for specific research in spatial cognition, and we will develop new programs for a large variety of relatively unexplored directions in mathematical papercrafting as well.

In outline, our research will include the following elements:

- Characterizing the nature of "spatial expertise" in the understanding of three-dimensional forms, and developing novel assessment techniques to measure development of that expertise;
- Devising a practical spatial curriculum (i.e., curricular activities designed to enhance and exercise spatial cognition) based on creative computational papercrafting activities;

• Extending our existing software environments to incorporate online *spatial advisors* that assist students in understanding and interpreting the polyhedral forms that they create, and that blend spatial exercises into more open-ended crafting activities; and
• Creating new software to extend the range of mathematical papercrafting activities in our curriculum.

The remainder of this report will elaborate on these elements. The following (third) section summarizes our own work in related NSF-sponsored projects. The fourth section discusses in turn each of the four research goals listed above. Finally, we include a timetable for our proposed work.

Figure 1. Top row: two HyperGami polyhedra—a variant of the regular icosahedron (left), and a great stellated dodecahedron (right). Bottom row: a compound of three cubes (left) and a family of polyhedral penguins (right).

3. Previous Related Work

Relevant prior NSF Research Grants.

Principal Investigator
National Science Foundation Young Investigator Award, August 1992-1997. (Award # IRI-9258684.) This is a five-year award for $25K/yr from NSF; in addition NSF matches up to $37.5K/yr of corporate support. [Total awarded: $282K/5 yrs (sum of $125K in "baseline" funds and $157K in matching funds).]

As Co-Principal Investigator
Grant #CDA-9616444. M. Resnick (MIT), R. Berg (Wellesley), S. Turkle (MIT), and M. Eisenberg. "Beyond Black Boxes: Bringing Transparency and Aesthetics Back to Scientific Instruments."
Amount: $880,658/3 years (Jan. 1, 1997-Dec. 31, 1999).

Grant #RED-9253425: G. Fischer, M. Eisenberg, and H. Eden: "Mastering High-Functionality Computer Systems by Supporting Learning on Demand", 1992-1995, amount: $1,504,238.

Grant #CDA-9408607: J. Spohrer et al.: "Next Generation Authoring Tools and Instructional Applications:, a collaborative grant of the Technology Reinvestment Program (TRP) jointly with Apple Computer, Carnegie-Mellon University, Stanford University, and the University of Massachusetts, Amherst, 1994-1996, $414,235.

Grant #REC-9553771: G. Fischer, M. Eisenberg, A. Repenning, H. Eden: "Learning by Design: Environments to Support Reinventing and Reengineering Education as a Lifelong Process", 1995-1996, amount: $398, 482.

Grant #REC-9631396: G. Fischer, M. Eisenberg, A. Repenning, H. Eden: "Lifelong Learning—Bringing Learning Activities to Life", 1996-1999, amount: $1,935,996.

The primary contributions of the co-PIs to these grants have centered around the themes of creating design environments for children; investigating the role of end-user programming in application design; integrating computational and craft activities; and creating "computationally-enriched" physical objects and artifacts for education by endowing craft objects with embedded, end-user-programmable computers. For the purposes of this proposal, summaries of our current and recent research may be surveyed through the following websites, as well as through the references listed in the biographical summaries appended to this project.

For craft technology work: www.cs.colorado.edu/~ctg/
For HyperGami/JavaGami in particular: www.cs.colorado.edu/~ctg/projects/hypergami/
Home pages of the PIs: www.cs.colorado.edu/~duck/ and www.cs.colorado.edu/~eisenbea/

Institutional Support.
For this proposal, the co-PIs will be able to make extensive use of the many institutional resources available to them. The first PI is a member of the Computer Science Department of the University of Colorado at Boulder, as well as the University's Institute of Cognitive Science (W. Kintsch, director) and the Center for Lifelong Learning and Design (G. Fischer, director). The second PI received her doctorate in Computer Science from the University of Colorado, Boulder. Both PIs have done extensive work with students at the New Vista High School in Boulder (R. Wilensky, principal) and have conducted workshops at the Collage Children's Museum in Boulder. New Vista and the Collage Museum will both be sites where the software and activities described in this proposal will be implemented and assessed.

4. Research Themes

In this section, we elaborate in turn on each of the four research elements described at the close of section 2.

4.1 Characterizing and assessing spatial "expertise"

The quotes from Ulam and Geller with which we began suggest that both these scientists have an informal notion of what it means to be a "spatial expert"—of what distinguishes a successful from a less successful spatial thinker. Indeed, the history of science is graced by a number of memorably brilliant spatial thinkers such as Kepler, Poincare, and Einstein. It is of especial interest, then, to try to understand and flesh out precisely what skills these thinkers possessed and to assess those skills with an eye toward pedagogical intervention.

One clue to answering such questions may be gleaned by examining the writings (and most interestingly, the diagrams and drawings) of powerful spatial reasoners. Consider, for instance, the following sketch of an icosahedron from Kepler's book *Harmonices Mundi*:

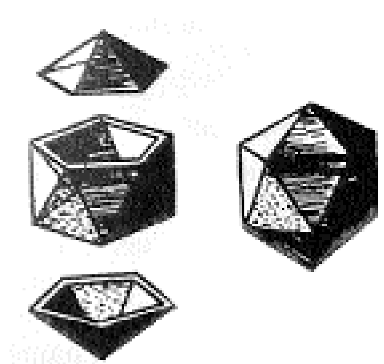

Figure 2. Kepler's interpretation of the icosahedron. (Originally in Kepler's *Harmonices Mundi*; reprinted in Cromwell [1997].)

For our purposes, the interesting aspect of this sketch is that it indicates one means by which Kepler was able to understand, to "parse", an otherwise potentially difficult shape. Apparently, Kepler was here reinterpreting the icosahedron as a (vertically aligned) collection of three distinct shapes: two pentagonal pyramidal "caps" on either side of a pentagonal antiprism. This insight not only renders the icosahedron itself more accessible; it also suggests a general recipe for potentially interesting shapes (e.g., we might now wonder whether a square antiprism could be similarly "sandwiched" between pyramidal caps, and what sort of composite shape that might produce).

As a "geometric thinker", Kepler is not alone in his apparent technique of visualizing relationships between polyhedral forms. The following two diagrams, taken respectively from Critchlow[1969] and Holden[1971] indicate similar strategies for identifying and describing such relationships. Holden's diagram, for instance, provides an unforgettable demonstration of the dual relationship between the cube and octahedron.

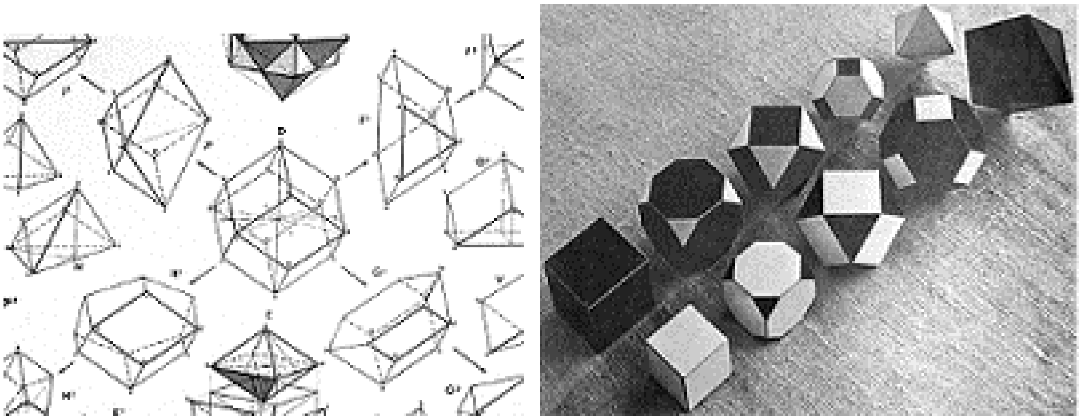

Figure 3. At left, a diagram from Critchlow [1969] showing a variety of operations on a rhombic dodecahedron. At right, a diagram from Holden [1971] in which two series of increasing truncations illustrate the dual relationship between the cube and octahedron

Diagrams such as these suggest potentially interesting new directions for characterizing and assessing spatial expertise. Currently, cognitive scientists do employ a variety of spatial reasoning tests, but most are relatively abstract and geared toward providing simple right-and-wrong answers (e.g., comparing two pictures of cubes to determine whether they could be views of one and the same solid [Ekstrom 1976]). We believe that these tests are an important part of the overall assessment process, but that they are not terribly fecund for educators: they lend themselves most naturally to simple drill-and-practice intervention (e.g., by showing students many practice examples of two-cube portraits). In contrast—and closer to the spirit of what spatial thinkers seem, in fact, to do—we believe that spatial expertise should be characterized and assessed by the presence of a variety of themes and heuristics similar to those popularized by Polya[1957] for general mathematics education. Thus, we hope to devise a collection of techniques for "how to see and interpret three-dimensional objects" analogous to Polya's "How to Solve It" heuristics; and we plan to accompany these heuristics with practical and flexible assessment techniques.

The sorts of heuristics (and related assessment techniques) that we have in mind include the following:

• *Fluency in characterizing and reinterpreting shapes*. A powerful spatial thinker should be able to interpret shapes in a variety of ways (including, potentially, playful or artistic ways). Following Kepler's example, an icosahedron might be interpreted as a bicapped pentagonal antiprism; or alternatively, it might be interpreted as a set of three "golden-ratio rectangles" with edges between nearby vertices; or it might be interpreted as the convex hull of the face midpoints of a dodecahedron; or it might be interpreted as a biological model of a virus; or as a wizard's crystal ball. (See Figure 4.) One way to assess spatial expertise, then, would be to find a variety of plausible metrics for students' fluency in characterizing and reinterpreting shapes, both in mathematical and nonmathematical ways.

• *Developing a vocabulary of symmetry groups*. One of the most mathematically fertile ways of thinking about polyhedral forms is through their associated symmetry groups. Typically, group theory is regarded as a topic in advanced mathematics; but polyhedra, through their very appeal, embody an introduction to symmetry operations—in Papert's phrase, they are excellent "objects to think with". We believe that there is an opportunity being missed here. Perhaps the major current obstacle to introducing younger students to group theory through polyhedra is that the vocabulary and terminology of symmetry groups is designed more for conciseness than pedagogical effectiveness. We propose to work with students in discussing, debugging, and communicating a variety of alternative notations for symmetry—an activity similar in spirit to diSessa's[2000] work in which elementary school children developed notations for the purpose of graphing motion. Our goal here is not to render all notations equal (or even equally acceptable), but to both stimulate and assess children's thinking about symmetry in three dimensions—to see how children, in the course of speaking through their ideas, might move from the vague intuition that there is *something* similar between the cube and octahedron to the eventual understanding that the two forms share identical types of symmetry. Moreover, a computational approach to polyhedral forms leads to still a different, procedural heuristic and language for understanding symmetry—a way more closely related to the manner in which symmetry actually arises, dynamically, in nature. In other words, a student who understands that a cube can be "chopped away" at each corner to form a cuboctahedron has begun to understand, in both a visual and operational way, that the symmetries of the two shapes must be the same.

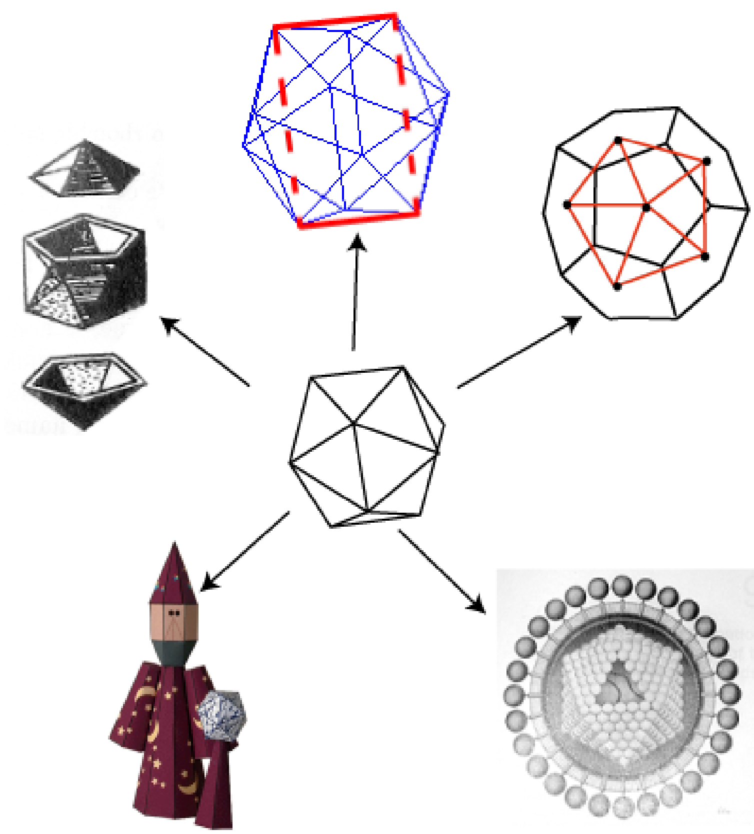

Figure 4. Interpretations of the icosahedron. Clockwise from upper left: Kepler's analysis of the shape; a "golden rectangle" hidden among four vertices of the shape; constructing the shape from the centers of faces of the regular dodecahedron; the HTLV-1 virus [Kappraff 1991]; a polyhedral wizard with an icosahedral crystal ball.

• *Craft, construction, and "tactile" heuristics*. Often, it appears, powerful spatial thinkers are able to understand forms not only as *mathematical* but also as *physical* objects—as objects physically constructed, sensed by touch, or emerging from natural processes. (These elements can be apprehended in the quotes from Ulam and Geller with which this document begins.) Such heuristics for understanding shape—which we might call construction, or craft, or tactile heuristics—are rarely discussed (much less assessed) in the mathematics education literature (though compare [Senechal 1990]). For instance, while a cube is, in some sense, a "stable" shape—it certainly is harder to roll along a surface than, say, an icosahedron—it is "unstable" in the sense that if it is constructed only out of dowels for edges, the structure can easily collapse. Out of such homespun insights, interesting mathematical and physical investigations may be formed: one might ask, for instance, how many additional reinforcement dowels (on diagonals of the cube's faces) might be needed to stabilize the structure.(cf. Kappraff [1991]). We propose to weave such construction and craft questions throughout our assessment techniques for spatial expertise—e.g., by asking children how certain types of shapes may be constructed, stabilized, stacked, reinforced, disassembled, and so forth.

• *Aesthetics and "relationship" heuristics*. In all our discussions of polyhedra as mathematical and physical forms, we never wish to lose sight of the reason that we (and, on happy occasions, our students) have come to enjoy them in the first place. Polyhedra can be beautiful, awe-inspiring, playful, and even humorous; and any true characterization and assessment of spatial thinking ought to respect this side of the subject. Spatial expertise, then, might well include the ability to create new types of polyhedra (through combining elements or pieces of others); or the ability to play with new uses for polyhedra (such as the gorgeous use of cuboctahedra for traditional lamps in some Japanese cities); or the ability to employ polyhedral elements in sculptural forms (such as the use of a dodecahedral head Winnie-the-Pooh in the sculpture below, done by a high school student).

Figure 5. A polyhedral Winnie-the-Pooh with a head based on the regular dodecahedron (designed and constructed by a high school student).

4.2 A spatial curriculum.

A major portion of our research effort will be devoted to working with K-12 students, with an emphasis on students at the middle and high school levels; to this end, we intend to continue our close working collaboration with the New Vista High School (a public school in Boulder) and with Boulder's Collage Children's Museum. Our goal is to create a body of activity-based curricular and assessment materials that will be made available to educators over the World Wide Web.

These materials will be tightly integrated with the techniques for characterizing and assessing spatial expertise discussed in the previous section; but they will, in the main, be conceived as activities, design projects, and games rather than as training exercises. For instance, one project might be to design a display similar to Holden's diagram, showing how one shape may be smoothly transformed to another. Another possibility might be to create a game played on a three-dimensional board; or to design a surface decoration that will expressly highlight some symmetry of the shape on which it is used. Moreover, beyond these "assessment-type" elements, our materials will also include units on polyhedral operations such as vertex truncation; group or classroom projects, such as the creation of crystal models and "three-dimensional murals"; artistic projects such as the creation of mathematical sculptures and the identification of polyhedral forms in nature; and advanced topics such as the design of compound polyhedra [Holden 1971], and an introduction to ideas from differential geometry by drawing paths on surfaces. [Abelson and diSessa, 1980] Our spatial curriculum will be closely and extensively linked with topics from the traditional mathematics and science curricula, including: the structure of crystals; varieties of molecular symmetries in chemistry; introductory concepts from group theory; and polyhedral forms in biology (e.g., viral forms, or spatial arrangements of plant cells). Finally, our assessment materials and our findings on spatial heuristics will be accompanied by web pages that teachers and students may download free of charge; and we will design these pages so that people who (e.g.) are unfamiliar with HyperGami or JavaGami, or who wish to work only with "low-tech" polyhedral modelling, will still find valuable resources at our site. Thus, our curricular materials will be created with the explicit goal of a wide educational impact, complementary to but emphatically not dependent upon our particular research software.

4.3 Spatial advisors realized in software.

The current HyperGami and JavaGami environments are pure, open-ended design tools, in which students create and decorate new forms; thus, the software itself offers little in the way of explicit guidance or support for exercising the types of spatial heuristics alluded to in the paragraphs above. We plan to extend our software environments with online *advisors* that can be invoked by students to (a) suggest ways of altering solids to create newly-customized forms; (b) point out meaningful relationships between solids; and (c) suggest spatial exercises or puzzles related to the forms the student has created. (Cf. [Eisenberg and Eisenberg 1998b; Eisenberg, Nishioka, and Schreiner 1997b]) These advisors are similar in spirit to the *critics* championed by Fischer and his colleagues [1991], and to even earlier techniques for "coaching" [Burton and Brown 1982], in that they simply observe the student's activity and do not set the student's overall goals. However, unlike critics or coaches, the purpose of these advisors is not to evaluate the student's creation (which would generally be inappropriate for this rather free-form polyhedral design); rather, the intent is to enrich design activities with analogs to the phases of thoughtful reflection or self-explanation advocated for some problem-solving domains.[cf. Chi and Van Lehn, 1991; Schoenfeld 1985] Again, while these advisors will be incorporated into our own software, their design and operation should be entirely generalizable to a variety of software environments; one could, for example, adapt these advisors to general or professional tools for three-dimensional modelling and rendering.

Just to provide a brief illustration of how such a spatial advisor might work in practice, we could imagine a scenario in which a student has begun their polyhedral design work with an icosahedron, and has paused to consider what sorts of variations on this shape might be explored. By selecting a menu choice in HyperGami, the student could invoke spatial advisors that would suggest and diagram (among other possibilities) the notion of "slicing" an icosahedron through the plane defined by two parallel edges; or the notion of slicing the shape through the plane defined by a pentagonal set of linked edges (this latter operation is the one noticed by Kepler); or the idea of looking for particular symmetry operations (such as rotation through an axis defined by a vertex and the center of the shape) that might spark still other possible explorations. Figure 6 provides a rough idea of how this spatial advisor capability might be realized in the HyperGami interface: here, the software presents a spatial advisor panel which includes the original shape as it currently exists under construction; a selection of possible operations to try on the shape; and a view of the results of these operations.

For our part, creating such three-dimensional advisors in software represents a challenging mixture of computational geometry (e.g., in searching for symmetry operations upon solids, enumerating interesting "slicing planes", etc.), interface design (e.g., in representing symmetry properties, or interesting relationships between solids), and cognitive science (e.g., in determining which types of "spatial advice" are useful for students in particular contexts). We expect that research in this direction will be an especially useful illustration of both the difficulties and opportunities involved in integrating basic cognitive science research with pedagogically useful system design.

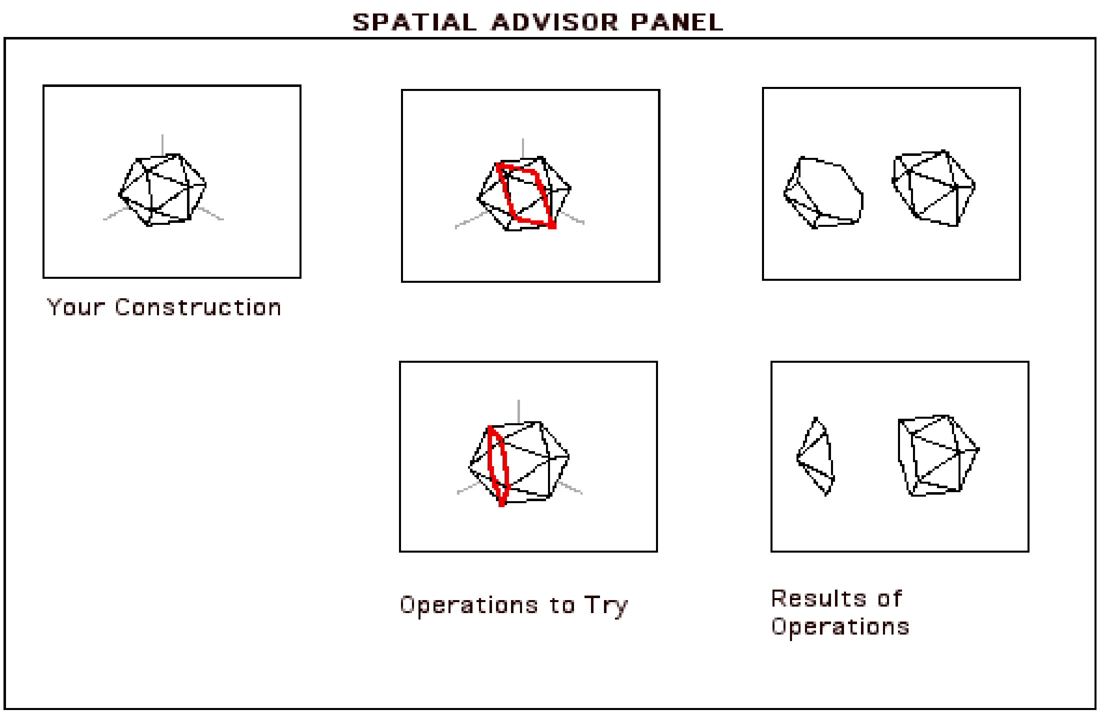

Figure 6. An idea of what the proposed software "spatial advisor" would look like. In this scenario, the user—who has begun by constructing a simple icosahedron shape—brings up a spatial advisor panel. The spatial advisor examines the icosahedron and finds two potentially interesting planes through which to slice the shape. (Note that the bottom example slices the plane in a manner similar to Kepler's parsing of the shape.)

4.4 New directions in computational papercrafting.

There are several crucial aspects to the design of the HyperGami and JavaGami programs that, we believe, afford students new opportunities for expressiveness within the traditional realm of polyhedral modelling. First, unlike traditional cut-out templates or kits, these programs allow students to construct a wider and more complex variety of shapes, and to customize (through various geometric operations) the shapes that they create. That is, students can not only create a huge range of classical shapes; they can (among other possibilities) slice, stretch, combine, or add new vertices to those shapes. (For one example, consider the heads of the polyhedral penguins shown earlier: these are created by adding a new vertex to "cap" one of the faces of a classical dodecahedron.) Second, because HyperGami and JavaGami are computational media, they can exploit, "for free", the marvelous range of supporting software and infrastructure that the computer provides. HyperGami folding nets may be decorated by tools within the program itself; but they can also be exported to complex graphics applications such as Canvas and Photoshop for still more extensive decoration. In a similar vein, folding nets may be sent by email, embedded within reports, or placed on the World Wide Web. In short, many of the advantages of computational papercrafting arise because of the surrounding "culture of applications" that is already, more or less, in place. Third—and in our view, most important—computer applications act as carriers of new notations, formalisms, and languages for thinking about mathematical crafts. This is an element that is truly lacking in all curricular materials of which we are aware, but it is implicit in a diagram such as the one by Critchlow shown earlier, which suggests a variety of unnamed "operations" that may be performed upon a basic shape. A computer program gives these operations names and algebraic (compositional) structure. To take one example, a starting prism *P* may be capped, stretched along one axis, and (finally) the new cap may be truncated to produce the shape *P'*: and the idea may be named and exercised on *any* starting prism P (and on many other shapes besides). Computer applications such as HyperGami and JavaGami encourage students (and designers) to regard mathematical notations as interesting artifacts in their own right; indeed, the former application includes its own end-user programming language (based upon the Scheme language) that allows advanced students and professionals to express extremely complex geometric ideas.

The purpose of this discussion is to suggest that there is a vast landscape of mathematical crafts—marvelous, artistic, (in a few cases) forgotten, and (in a few other cases) untried craft activities—that could similarly benefit from the advent of computational design media. HyperGami and JavaGami provide an example of a domain in which hands-on work with tangible materials can be enriched by, and combined with, the potentialities of computational media. But the domain of polyhedral modelling is hardly unique in this respect. The mathematical range of paper—and related affordable materials, such as transparency film—is immense; and our enjoyable experiences to date with HyperGami and JavaGami have encouraged us toward continued exploration of mathematical papercrafting.

Among the many possibilities here would be software environments for design of curved shapes (i.e., shapes whose surfaces are not composed of simple planar polygons, as is the case with standard polyhedra); design of dynamic, flexible, or pop-up structures in paper; design of three-dimensional materials for studying perception through optical illusions and anamorphic art; techniques for representing two-dimensional surfaces in space through paper (as described, for example, in Cundy and Rollett [1961, sec. 4.8.1]); and techniques for creating decorated topological models (such as Moebius strips and their cousins). Traditional examples of a few of these papercraft activities are shown in Figure 7.

Figure 7 suggests the rich possibilities that exist for the development of powerful software design environments. In some cases (e.g., pop-up design), such environments for students would represent an introduction not only to mathematical and geometric ideas, but also to some interesting homespun mechanics and engineering. In others (e.g., the design of optical illusions and anamorphic art), a powerful design environment could spark explorations into creative experimental psychology. In still others (surface models), complex and beautiful mathematical models of a sort very different from polyhedra could be (e.g., models of two-dimensional potential functions) could be created and explored. To date, very little of this astonishing landscape has been touched by educational technology; we believe that powerful software environments could revive and extend these traditional (and generally underutilized) crafts. Again, it is worth emphasizing that the creation of such environments would neither supplant nor trivialize students' work with physical materials, but would on the contrary be focused upon enriching and extending this work. A student using (e.g.) a "pop-up design environment" would do so as one stage of a process that would culminate in a beautiful and fascinating physical object (in much the same way that a HyperGami user designs a shape that eventually is realized as a tangible form in paper).

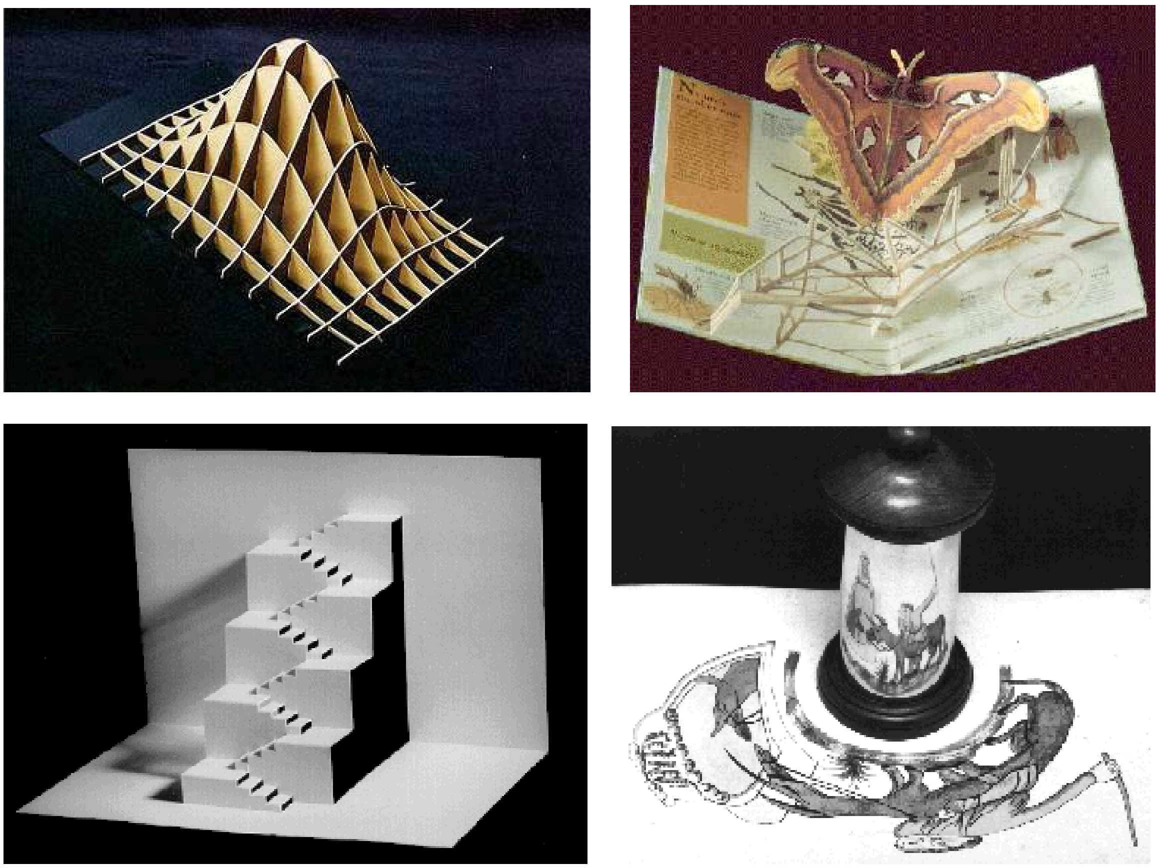

Figure 7. Examples of traditional mathematical papercrafts that suggest the expressive possibilities of computational design environments. Top left: A "sliceform" realization of a curved surface. Top right and bottom left: Pop-ups. Bottom right: Anamorphic art in which a distorted image is reflected "normally" in a cylindrical mirror. (Images from the World Wide Web.) In each of these cases, one could envision a design environment for students that would extend the complexity, mathematical/scientific content, and aesthetic expressiveness of students' hands-on activities.

Timetable for Proposed Work:

Year 1: Initial development of spatial curriculum and assessment materials; development and working implementation of spatial advisor elements. First round of testing and evaluation with high-school students at New Vista; workshops with elementary-school-age children at Collage Children's Museum. Initial work on software system development for at least two additional domains in mathematical papercrafting.

Year 2: Revision and continuing development and assessment (at New Vista and Collage) of curriculum and assessment materials. First versions of materials to be made available on the World Wide Web. Working prototype development of additional software. Continuing development of spatial advisors.

Year 3: Final revisions and publications (on the Web and in print) of curriculum and assessment materials. Specifications (non-software-specific) for spatial advisors to be published and made available for use in other modelling systems. Completion of working systems for additional mathematical crafts. Evaluation of new working systems at New Vista and Collage.