Mathematical String Sculptures: a Case Study in Computationally-Enhanced Mathematical Crafts

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1. Introduction: The Space Spider

In the 1960's, a lovely toy called *Space Spider* [P2] introduced children to the design of mathematical string sculptures. Space Spider was relatively simple in its design: I'll describe it in a moment, but the reader can get a good initial sense of the toy by looking at Figure 1, which shows a Space Spider construction from the original documentation. [6]

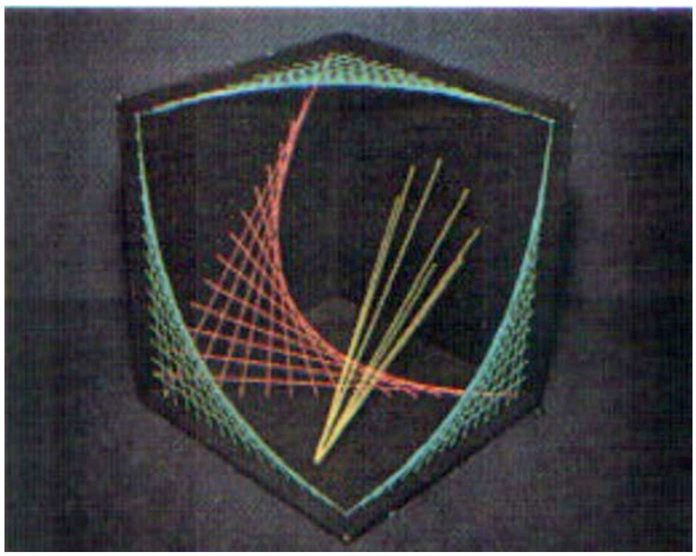

Figure 1. A Space Spider construction with four parabolic curves and a "spotlight", from the original documentation for the toy. [6]

The black frame in Figure 1 was constructed from three cardboard pieces. Each of these pieces was square-like in shape, but had ridged edges that permitted the given piece to fit into edges on the other two. Once locked together, the pieces formed an octant in space; one could thus think of the three squares as representing the *xy*-, *xz*-, and *yz*-planes. The pieces also had arrays of labeled holes through which thread could be inserted.

Besides the three cardboard pieces, the Space Spider toy also included rolls of brightly colored elastic thread and documentation that described "recipes" for the construction of particular models. Once the three-piece frame was put together, the child could weave the thread in and out of the holes in the frame to create lovely patterns–approximations of mathematical curves and surfaces. The recipes themselves were written in a kind of shorthand notation that told the child which holes to use when weaving the thread: for example, a pattern such as "X-A1-->X-B1-->Y-G10-->Y-G9-->..." specified that the thread should be placed through the A1 hole on the "X piece" (i.e., the *yz*-plane), then should re-emerge through the B1 hole on that same piece, then inserted into the G10 hole on the Y-piece (i.e., the *xz*-plane), and so forth. In the Figure 1 construction, five separate lines of thread (in three distinct colors) have been used to create the overall design.

2. A Brief Mathematical Interlude

The major purpose of this snapshot is to explore ways in which computational tools and output devices can enrich the practice of "mathematical stringcraft"–the sort of activity exemplified by the Space Spider toy. But before embarking on this discussion, it is worth pausing at the outset to connect the craft of string sculpture with more traditionally-presented mathematical ideas. This is not the occasion for a thorough discussion of using string to illustrate mathematical figures; still, we can at least describe the mathematics behind one straightforward example. (For those interested in going further, Millington's beautiful book includes numerous sample projects, mostly in two dimensions [4], while a straightforward mathematics text on curves such as [1] will often include figures that suggest string constructions of well-known forms.)

The photograph shown in Figure 1 includes four parabolic curves (three in blue, one in red) represented in string; more accurately, the string segments delimit the outline of an approximation to a true parabola. It is not immediately apparent that this is in fact the case–looking at Figure 1, one might wonder if the curves shown could actually be (say) fourth-order polynomials, or sections of hyperbolas. What follows, then, is a brief proof (or sketch of a more thorough proof) that the Figure 1 curves are in fact parabolic.

To simplify matters, we will focus on the three blue curves in Figure 1. These curves are generated by taking two lines at right angles–we'll begin by using the x- and y-axes–and joining string segments between pairs of holes at equal intervals for positive values of x and y, as shown in Figure 2. The point nearest the origin on the x-axis is joined to some chosen distant point on the y-axis; progressively further points on the x-axis are then joined to points approaching the origin on the y-axis. (A glance at Figure 1 will show that the three blue curves are in fact formed in just this way.)

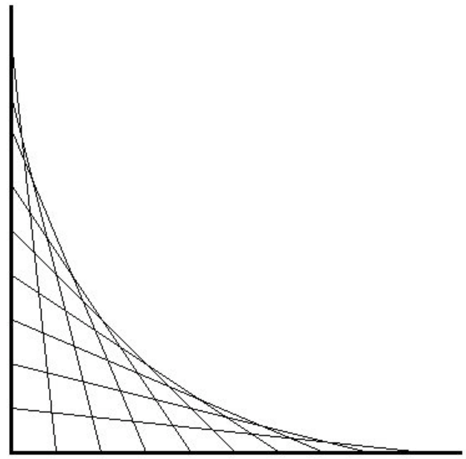

Figure 2. Joining string segments between evenly spaced locations (corresponding to holes) on the x- and y-axes. The location closest to the origin on the x-axis is joined to some particular distant location on the y-axis; as locations move further out to the right on the x-axis, they are joined to successively lower locations on the y-axis.

We wish to show that the intersection points of successive string segments lie along a parabola. For this purpose, we'll turn the two axes counterclockwise by 45 degrees, and choose our length scale so that the holes occur at the points $(\pm j, j)$. We will also, for simplicity, stipulate that the point (-1, 1) is joined to a point (j, j) where j is an odd number, written as $2k$-1; thus the situation looks as in Figure 3. Note that in Figure 3 we have a horizontal line between the points $(-k, k)$ and (k, k); this line is not quite tangent to the parabola at its minimum point, but it seems to come pretty close to that description. Note also that the intersection points have reflective symmetry about the y-axis; thus, if the intersection points labeled in Figure 3 occur on a parabola, the equation of that parabola will be of the form $y = ax^2 + c$.

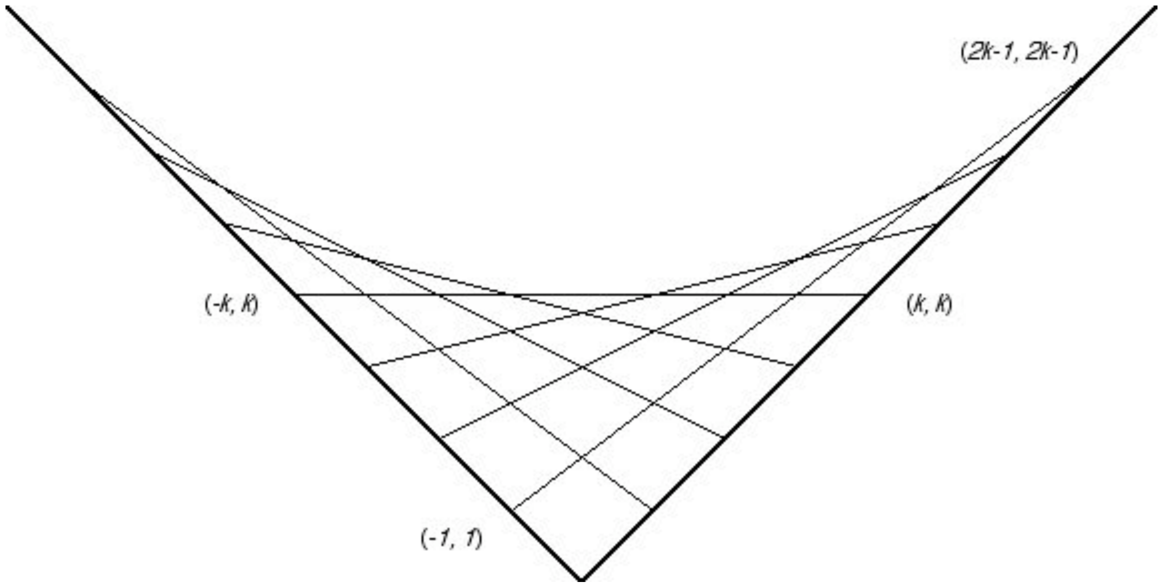

Figure 3. Here, we have drawn string segments between the lines $y = x$ and $y = -x$. (In effect, we're rotating the axes of Figure 2.) The point (-1, 1) is joined to the point (2k-1, 2k-1); the point (-2, 2) is joined to the point (2k-2, 2k-2); and so forth. Note that we have a horizontal string segment between *(-k, k)* and (*k, k*). Our goal is to show that the intersection points between successive string segments lie along a parabola. (For this particular drawing, k has a value of 4; but clearly the same type of figure could be created for any value of k greater than 1.)

From here, it is a matter of fiddling with algebra to show our result. We observe that each string line joins two points of the form ($-k+m$, $k-m$) and ($k+m$, $k+m$) for integers m in the range ($-k$+1, k-1). The equation of the line between two such points (for a chosen value of m) is

$$y = (m/k)\, x + (k - (m^2/k))$$

The intersection point between the lines corresponding to integers m and m+1 can now be determined as: $(2m+1,\ (k^2 + m^2+m)/k)$

And finally, using these successive intersection points to determine our parabola, we find that the equation of the parabola is

$$y = (x^2/4k) + (4k^2-1)/4k$$

In other words, the constants a and c in our parabola, are $1/4k$ and $(4k^2-1)/4k$, respectively.

To take a specific example: suppose we join the point (-1, 1) to (9, 9) by string; then (-2, 2) to (8, 8); (-3, 3) to (7, 7); and so forth. Then our value for k is 5, and the parabola determined by the intersection points is

$$y = 0.05x^2 + 4.95$$

From this, it should (I hope) be plausible that similar string-intersections between equally spaced points on two lines form a parabola if the value of j above is even, and likewise if the angle between the two lines is not a right angle (this is the case for the curve formed by red string in Figure 1). A particularly diligent reader might try to accomplish something that I struggled with unsuccessfully for several days–namely, to show a straightforward relationship between the intersection points of the string segments and the focus and directrix of the resulting parabola.

3. Re-fabricating the Space Spider

Having detoured into the mathematics of string figures, we can now return to Space Spider itself. In its time, Space Spider was illustrative of the burgeoning mid-century (or perhaps post-Sputnik) interest in creative mathematics and science education in the United States. Regrettably, the toy is now no longer available, though the occasional preserved "vintage" artifact occasionally shows up for sale on the World Wide Web. The remainder of this snapshot is an attempt to alleviate this problem by showing how–with the aid of a computer and laser cutter–we can now recreate and even extend the original capabilities of the Space Spider toy.

For those unfamiliar with the device, a brief description of the laser cutter might be in order here. Basically, the laser cutter is a computer-controlled output device that employs a moving laser beam to cut through (or, at lower power, etch lines into) materials such as wood, acrylic, and cardstock. The high precision of the device makes it remarkably versatile: indeed, laser cutters can be used for a huge range of mathematical crafts. (Cf. [2]) For our purposes, by using the laser cutter to slice the outlines and holes of Space-Spider-like frame pieces in basswood, we can recreate our own model of the original toy. Figure 2 shows three wooden octants that have been constructed from nine laser-cut pieces; the octants are then used as the frames for string sculptures. The sculpture at left was constructed with elastic thread, while the two toward the right were made with standard knitting yarn.

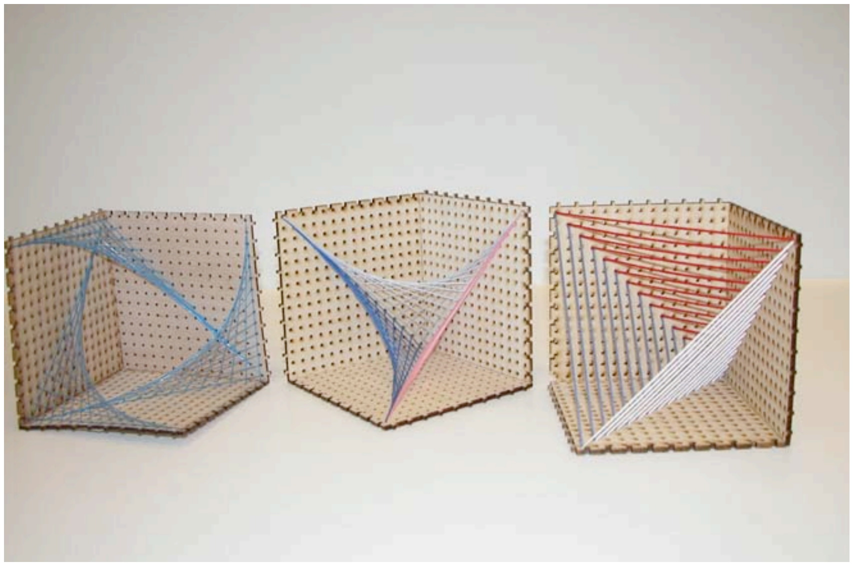

Figure 4. Three Space Spider constructions produced with laser-cut basswood frames. The two at left include curve designs similar in construction to those in Figure 1; at right, the strings suggest the interior of a regular tetrahedron.

The purpose of Figure 4 is just to show that we can essentially recreate the standard Space Spider toy. Admittedly, a couple of details have been changed: for example, the original toy had small labels for the rows and columns of the pieces to assist the user in finding the individual holes given in the construction recipes. The pieces in Figure 4 do not in fact have these labels, though it wouldn't be hard to include them through the use of laser-etched lines on the surface of the wood. Also, the basswood pieces have a light coloring (I prefer this), though again it would be possible to use darker wood, or to apply paint to the laser-cut pieces, or even to make the frame pieces using acrylic of any desired color. In Figure 4, the laser-cut holes in each piece are in fact tiny squares, and are cut at regular intervals in both the x- and y-dimensions.

In a sense, Figure 4 already suggests one way in which computationally-enriched crafts take us beyond the original toy: namely, by purchasing the necessary wooden pieces at a local hobby shop, we can then use the laser cutter to make as many frames as we wish. The original toy came with a limited number of frame pieces, which meant in turn that a child's earlier string sculptures had to be disassembled in order to make new ones. Now, because we can make as many frames as we wish, our constructions can be displayed permanently without the need for reclaiming frame pieces. This may seem like a relatively small point, but in my own view it has a tremendous impact on the way in which one thinks about the craft of string sculpture: because craft constructions can now be built without fear of disassembly, we can give them away as gifts, compare a family of constructions side-by-side, or put our constructions on the shelf for as long as we wish. Moreover, since the frame specifications are stored as computer files that can be printed out on the laser cutter, we can (e.g.) make frames at various scales and in a variety of distinct materials.

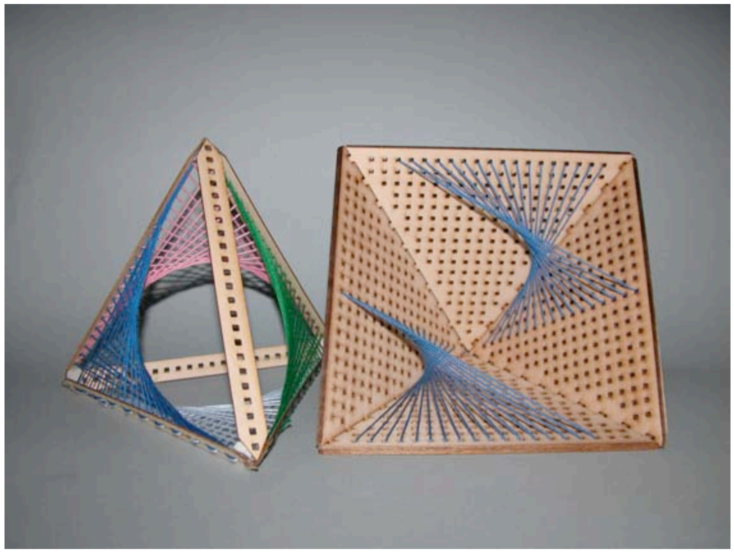

Figure 5. A string sculpture produced in a tetrahedral frame (at left), and in the top half of an octahedron (at right).

3. Space Spider Variations

The examples of the previous section represent only an initial step, however: now that we have crossed the Rubicon of recreating the original toy, it is a natural progression to employ the laser cutter in ways that go beyond the original. To take one possibility: we needn't restrict ourselves to frames that bound an octant in space. By creating four frame-pieces in the form of ridged equilateral triangles (instead of squares, as in Space Spider) we can create a frame that is the open bottom half of an octahedron, as shown toward the right of Figure 5. (And there are still other possibilities for "open-polyhedron" frames that I haven't tried–e.g., six pentagonal faces of a regular dodecahedron might make an attractive frame as well.) Toward the left of Figure 5 we have cut six slender "edge-frames", each with a row of holes, and arranged them into a tetrahedron: the tips of the edge pieces are cemented together on four small paper tetrahedra to form the vertices of the overall frame. The string figures in this particular case lie along the planes of the faces of the overall tetrahedron, though it would also be possible to create sculptures in which string passes through the interior of the frame. (This same design can be found in a fine book by Pohl [5], in which straws are used as the basic material of the frame structure.)

Yet another possibility is to create circular frames through which string can be threaded. (In fact, to give credit where it's due, the original Space Spider toy could also be purchased with a set of circular metal "Space Rings" for this type of construction.) In Figure 6, two wooden circular rings have been laser cut; the two rings contain small regularly-spaced holes for thread, and three larger holes in which acrylic struts may be placed. The resulting frame construction provides a foundation upon which to create a model of a hyperboloid.

Figure 6. A string hyperboloid produced by weaving string between two parallel ring frames. The rings are separated and supported by three clear acrylic dowels.

Figure 6 suggests still another genre of string constructions in which string is threaded through holes in two parallel planes. This idea is pursued still further in Figure 7. Here, rather than having two circular arrays of holes in the two parallel planes (as in the Figure 6 design), we have tried still other patterns of thread-holes in two opposite sides of an open box. The construction at left in Figure 7 is similar to the one in Figure 6, except that the sequences of holes in the two opposite sides form identical ellipses rather than circles. The center construction also employs two elliptical patterns of holes, except in this case the major and minor axes of the top ellipse are 90 degrees offset from those of the bottom ellipse. The figure at right, constructed by Kaylin Spitz, makes use of two parabolic arrays of holes: here, the curve of the parabola at the top faces in the opposite direction from that at the bottom. (The directrix of the top parabola is thus parallel to that of the bottom parabola.)

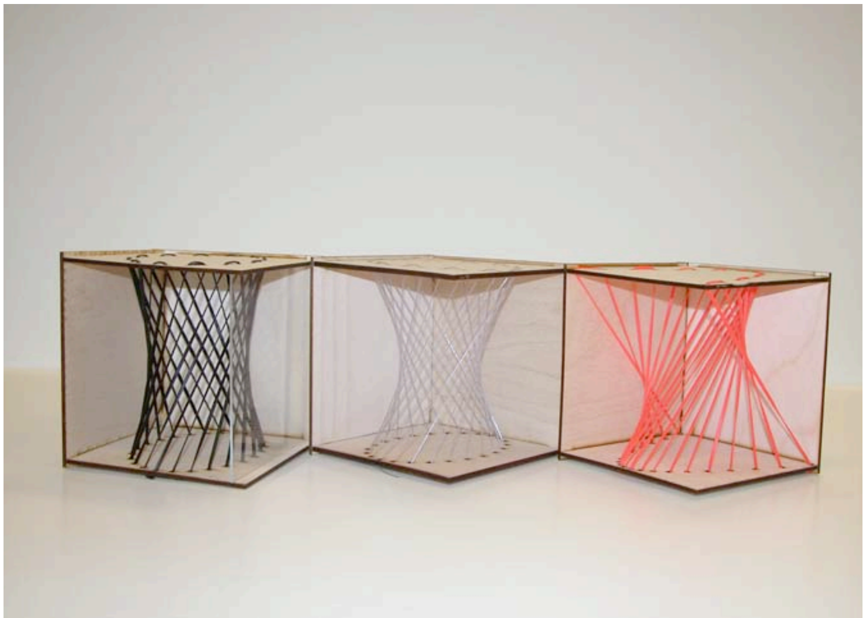

Figure 7. Three string sculptures created by weaving string between two parallel planes. At left, the two planes have holes placed around two ellipses; at center, the two ellipses in the two planes have their major and minor axes at right angles to each other; at right, the two sets of holes are placed along parabolas in opposite orientations.

It is, of course, a bit of a limitation that the laser-cut surfaces from which our frame holes are created are all flat. Standard laser cutters are primarily used to cut flat surfaces (there are exceptions, but this is not a detour worth exploring for our purposes). Still, one can attempt to finesse this issue in several imaginable ways. One intriguing possibility is to laser-cut a flat wooden piece and then bend the piece: a specially treated commercially-available material called Bendywood [P1] can be used for this purpose. Bendywood is steam-treated wood that can be bent (not without some effort!) into curved shapes; as one might expect, the wood is not infinitely pliable, but it can be curved into an arch as shown in Figure 8. Thus a set of laser-cut holes in a single piece of Bendywood can be used to create a string sculpture. In Figure 8, the holes are cut in three distinct circular patterns along the length of the Bendywood piece; the wood is then bent and fixed in an acrylic base; and the string pattern is threaded through the holes of the (now-bent) single piece.

4. Further Exploration in Technologically-Enhanced String Sculpture

The examples shown in this snapshot are themselves merely initial explorations of the ways in which mathematical string crafts, in the spirit of the original Space Spider toy, can be extended. Just to mention a few possibilities:

1. To return to Figure 4 for the moment: the laser-cut holes in the frame pieces of these constructions are placed at the vertices of a regular square array. (This was true of the original toy as well.) It would be an interesting experiment to create alternative arrays of holes–e.g., holes placed at the vertices of a triangular tiling, or at the vertices of a

semiregular tiling, or holes placed with certain degrees of randomness, or holes specifically placed to produce a particular string sculpture that would be unachievable using the original toy. The same basic idea could also be applied to the frames of Figure 5, or to the ring-based holes of Figure 6: we needn't space the holes at identical regular intervals around the two circles of Figure 6, for instance, but could try placing them at every nine degrees in one half of each circle and every five degrees in the other half. Or, to take another possibility, we could place forty holes at nine-degree intervals all around the top circle, and thirty holes at twelve-degree intervals all around the bottom circle: in this case, the sets of holes are regularly spaced, but the two circular frames employ different spacings.

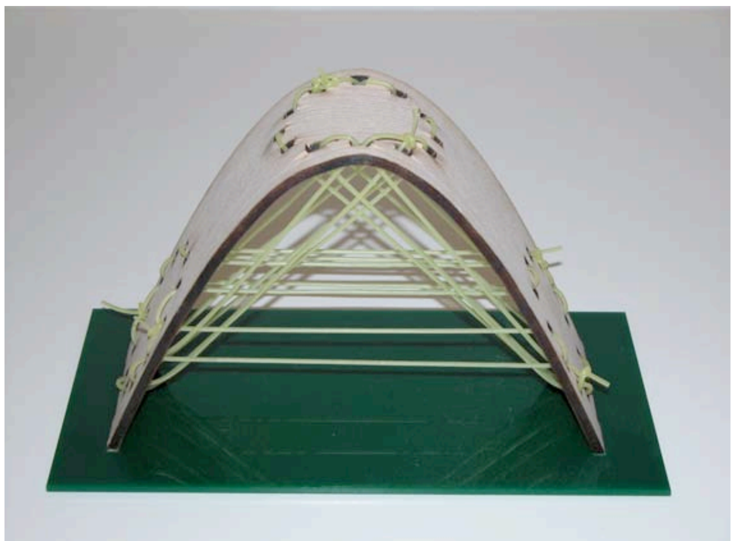

Figure 8. A string sculpture woven into a single-piece "Bendywood" arch.

2. A natural step for exploration would be to try alternative materials for construction. We have already mentioned the possibility of using laser-cut acrylic for frames; still other experimentation might focus on the string elements of the constructions. For example, electroluminescent wire (string that lights up in a bright color when a voltage is applied across it) might be used for portions of string constructions; or conceivably LED lights could be spaced along a length of conductive thread; or one might try employing strips of decorated elastic fabric, rather than string, for certain portions of constructions. One particularly appealing project might be to experiment with shape memory alloys: these are often sold commercially in the form of "wires" that could be incorporated within string sculptures. A string sculpture with a shape memory alloy element might incorporate some interesting dynamic effects (e.g., a piece of shape memory wire might be used to bend or alter the supporting frame of a sculpture under computer control, thus changing the arrangement or tension among the strings).

3. Going beyond the laser cutter, one could imagine using a 3D prototyper (or "3D printer") to create specialized frames for string sculptures. Thus, one might weave strings through holes embedded into (e.g.) helical surfaces, Moebius-strip-like surfaces, and

innumerable other possibilities. This would be a natural response to the limitations of the laser cutter as a frame-making device–though the earlier examples should suggest that there is plenty of experimentation to do with the laser cutter alone.

4. Finally, it should be mentioned that there is tremendous scope for exploration of software designs to enhance the practice of mathematical string sculpture. An early project in our lab, by T. Chen, involved the design of a program called "HyperSpider" whose purpose was to help students experiment with Space Spider-like designs on the computer screen before actually building them with physical materials. A screen shot of HyperSpider is shown in Figure 9: the basic idea behind the program was that the user could weave "virtual string" in and out of the holes of the frames shown, and could view a 3D rendering indicating how the thread pattern would look once built.

HyperSpider is described at somewhat greater length in earlier publications [2, 3], and in any event it never got to the (much-desired) stage of being a widely releasable, robust application; but it does hint at the marvelous possibilities for software enhancement of mathematical string crafts. One might, for example, use an enhanced HyperSpider-like tool not only to experiment with string patterns in some particular frame, but to experiment (in the spirit of our earlier variations) with new frame structures as well. This sort of development would represent the ultimate goal for computationally-enhanced crafting, in which creative software can vastly expand the range of mathematical creation in physical materials.

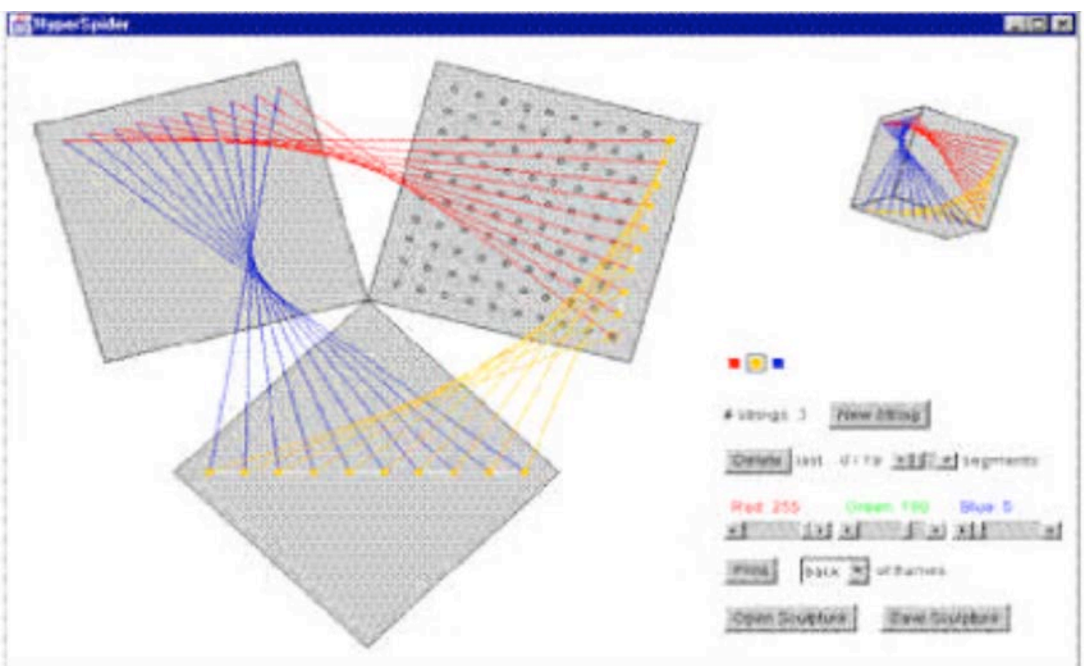

Figure 9. A screenshot of the HyperSpider program in the course of a construction scenario. Three threads (red, yellow, and blue) are being woven among the three "exploded" frames at left, and at right a 3D view of the thread pattern is visible. [2]

Acknowledgments. Thanks to Ann Eisenberg for the photographs in Figures 4-8. Kaylin Spitz built one of the Figure 7 constructions; Ted Chen designed and programmed the HyperSpider application. Gerhard Fischer, Mark Gross, Clayton Lewis, and Andee Rubin have been the instigators of especially wonderful conversation on the subject of this paper. The research described in this paper was partially funded by the National Science Foundation under awards no. EIA-0326054, and REC0125363.

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Products

[P1] Bendywood. www.bendywood.com

[P2] Space Spider [Space Materials Center] Space Materials Center. Space Geometrics Toy. New York.